

Exact solution of D_N type quantum Calogero model through a mapping to free harmonic oscillators

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Abstract

We solve the eigenvalue problem of the D_N type of Calogero model by mapping it to a set of decoupled quantum harmonic oscillators through a similarity transformation. In particular, we construct the eigenfunctions of this Calogero model from those of bosonic harmonic oscillators having either all even parity or all odd parity. It turns out that the eigenfunctions of this model are orthogonal with respect to a nontrivial inner product, which can be derived from the quasi-Hermiticity property of the corresponding conserved quantities.

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1 Introduction

Exactly solvable quantum many particle systems and spin chains with long-range interactions [1–10] have attracted much attention due to their appearance in apparently diverse branches of physics and mathematics like generalized exclusion statistics [9, 11–14], quantum Hall effect [15], quantum electric transport in mesoscopic systems [16, 17], random matrix theory [18], multivariate orthogonal polynomials [19–21] and Yangian quantum groups [22–24]. The study of this type of models with long-range interaction was initiated by Calogero [1], who has found the exact spectrum of an N -particle system on a line with two-body interactions inversely proportional to the square of their distances and subject to a confining harmonic potential. The Hamiltonian of such rational Calogero model may be written in the form [1, 2]

$$H_A = \frac{1}{2} \sum_{i=1}^N \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \alpha(\alpha - 1) \sum_{1 \leq i < j \leq N} \frac{1}{(x_i - x_j)^2}, \quad (1.1)$$

where α ($> \frac{1}{2}$) is a free parameter. It has been found that, this Hamiltonian yields a quantum integrable model associated with the A_{N-1} root system and it is possible to construct generalizations of this Hamiltonian for other root systems while preserving the quantum integrability property [4, 25, 26]. In particular, for the case of D_N root system, the Hamiltonian of Calogero model is given by

$$H_D = \frac{1}{2} \sum_{i=1}^N \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \nu(\nu - 1) \sum_{1 \leq i < j \leq N} \left[\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right], \quad (1.2)$$

where ν ($> \frac{1}{2}$) is a free parameter. Furthermore, the Hamiltonian of Calogero model associated with the B_N root system is related to its D_N counterpart as

$$H_B = H_D + \frac{1}{2} \rho(\rho - 1) \sum_{i=1}^N \frac{1}{x_i^2}, \quad (1.3)$$

where ρ ($> \frac{1}{2}$) is another free parameter corresponding to the one-body potential.

Due to Eq.(1.3), one may naively think that the D_N type of Calogero model is just a special case of its B_N counterpart and all physically relevant properties of the former model can be obtained from those of the latter model by simply taking the $\rho \rightarrow 0$ limit. However, the spectra of Calogero models associated with all root systems can be calculated by acting the corresponding Hamiltonians on Coxeter invariant Polynomials [25, 26]. It turns out that, contrary to the naive expectation, spectrum of the D_N type of Calogero model can not be reproduced from its B_N counterpart by taking the $\rho \rightarrow 0$ limit. Moreover, the spectra of BC_N and D_N types of Calogero models along with their spin generalizations have been computed recently by finding out appropriate sets of basis vectors on which the corresponding auxiliary Hamiltonians and Dunkl operators

act as some triangular matrices [27, 28]. Again it is found that, spectra of these D_N type of models can not be reproduced from their B_N counterparts as some special cases. Consequently, the D_N type of Calogero model and its spin generalization should be considered as some singular limits of their B_N counterparts.

Even though the eigenvalue problem of the D_N type of Calogero model (1.2) has been studied earlier through different approaches, the connection of the corresponding Hilbert space with that of free quantum harmonic oscillators (QHO) has not been explored till now. In this context it should be noted that, one can solve the eigenvalue problem of A_{N-1} and B_N type of Calogero models by using similarity transformations which map these models to a system of N number of decoupled QHO (up to some additive constants) [29–32]. However, due to the difference of domains on which these similarity transformations act as nonsingular operators, the spectrum of A_{N-1} type of Calogero model differs significantly from that of the B_N type of Calogero model. More precisely, up to a constant shift of all energy levels, the spectrum of the A_{N-1} type of Calogero model coincides with that of N number of bosonic QHO, which corresponds to completely symmetric wave functions [29, 32]. On the other hand, the spectrum of the B_N type of Calogero model can be identified with a subset of the spectrum of N number of bosonic QHO, which corresponds to completely symmetric as well as even parity wave functions [30, 32]. The orthogonality relations for the eigenfunctions of both A_{N-1} and B_N types of Calogero models have also been established [32]. The purpose of the present article is to make a connection between the Hilbert space of the D_N type of Calogero model (1.2) and that of QHO, by applying the method of similarity transformation.

The arrangement of this paper is as follows. In Sec.2, we describe the similarity transformation which maps this D_N type of Calogero model to a system of N number of decoupled QHO. In this section, we also find out the domain on which such similarity transformation acts as a nonsingular operator. By using these results, in Sec.3 we solve the eigenvalue problem of the D_N type of Calogero model. In particular, we construct the eigenfunctions of this Calogero model from those of bosonic harmonic oscillators having either all even parity or all odd parity. We also show that eigenfunctions of this model are orthogonal with respect to a nontrivial inner product, which has a close connection with the quasi-Hermiticity property of the corresponding conserved quantities. In Sec.4 we make some concluding remarks.

2 Similarity transformation and its domain

Our aim is to solve the eigenvalue problem given by

$$H_D \psi(x_1, \dots, x_N) = E \psi(x_1, \dots, x_N), \quad (2.1)$$

by constructing a similarity transformation which would map H_D (1.2) to a set of decoupled quantum harmonic oscillators (QHO). It is well known that, the ground state

wave function for the D_N -type Calogero model can be expressed as

$$\psi_0(x_1, \dots, x_N) = \prod_{1 \leq i < j \leq N} |x_i^2 - x_j^2|^\nu e^{-\frac{1}{2}\omega \sum_{i=1}^N x_i^2}, \quad (2.2)$$

and the ground state energy is given by

$$E_0 = \frac{1}{2}N\omega + \nu N(N-1)\omega. \quad (2.3)$$

By using the ‘operator form’ of this ground state wave function, we perform a similarity transformation on H_D as

$$\begin{aligned} \tilde{H}_D &= \psi_0^{-1}(H_D - E_0)\psi_0 \\ &= \sum_{i=1}^N \left(-\frac{1}{2} \frac{\partial^2}{\partial x_i^2} + \omega x_i \frac{\partial}{\partial x_i} \right) - 2\nu \sum_{1 \leq i < j \leq N} \frac{1}{(x_i^2 - x_j^2)} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right). \end{aligned} \quad (2.4)$$

Note that the eigenvalue equation (2.1) for H_D can equivalently be expressed as an eigenvalue equation for \tilde{H}_D :

$$\tilde{H}_D \phi(x_1, \dots, x_N) = (E - E_0) \phi(x_1, \dots, x_N), \quad (2.5)$$

where the corresponding eigenfunctions are related as

$$\psi(x_1, \dots, x_N) = \psi_0(x_1, \dots, x_N) \phi(x_1, \dots, x_N). \quad (2.6)$$

Let us now consider the Euler operator (O_E) and D_N type of Lassalet operator (O_L) given by

$$O_E = \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}, \quad O_L = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 4\nu \sum_{1 \leq i < j \leq N} \frac{1}{(x_i^2 - x_j^2)} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right), \quad (2.7)$$

which satisfy the commutation relation

$$[O_L, O_E] = 2O_L. \quad (2.8)$$

In terms of these two operators, \tilde{H}_D in Eq.(2.4) can be written in a compact form like

$$\tilde{H}_D = \omega O_E - \frac{1}{2} O_L. \quad (2.9)$$

By using the commutation relation (2.8) and the well known Baker-Campbell-Hausdorff (BCH) formula, we find that \tilde{H}_D (2.9) can be transformed into the Euler operator as

$$e^{\frac{1}{4\omega} O_L} \tilde{H}_D e^{-\frac{1}{4\omega} O_L} = \omega O_E. \quad (2.10)$$

Let us now define the Laplacian operator as $\nabla^2 \equiv \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$. It is easy to see that this Laplacian operator and Euler operator satisfy the commutation relation: $[\nabla^2, O_E] = 2\nabla^2$. By using this commutation relation and the BCH formula, one finds that

$$e^{-\frac{1}{4\omega}\nabla^2} (\omega O_E) e^{\frac{1}{4\omega}\nabla^2} = \omega O_E - \frac{1}{2}\nabla^2 \equiv \bar{H}. \quad (2.11)$$

Next, we introduce the operator $X^2 \equiv \sum_{i=1}^N x_i^2$, which satisfies the commutation relations

$$[O_E, X^2] = 2X^2, \quad [\nabla^2, X^2] = 2(2\hat{O}_E + N).$$

By using these commutation relations and the BCH formula, it is easy to find that

$$e^{-\frac{1}{2}\omega X^2} \bar{H} e^{\frac{1}{2}\omega X^2} = H_{QHO} - \frac{1}{2}N\omega, \quad (2.12)$$

where

$$H_{QHO} = \frac{1}{2} \sum_{j=1}^N \left(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2 \right), \quad (2.13)$$

represents the Hamiltonian of N number of decoupled QHO. Combining the relations (2.10), (2.11) and (2.12), we find that

$$T^{-1} \tilde{H}_D T = H_{QHO} - \frac{1}{2}N\omega \quad (2.14)$$

where

$$T = e^{-\frac{1}{4\omega}O_L} e^{\frac{1}{4\omega}\nabla^2} e^{\frac{1}{2}\omega X^2}. \quad (2.15)$$

Next, we try to construct the Hilbert space of Hamiltonian \tilde{H}_D from that of H_{QHO} , by using the similarity transformation (2.14). To this end, we consider the creation and annihilation operators of QHO given by

$$a_j = \frac{i}{\sqrt{2\omega}}(p_j - i\omega x_j), \quad a_j^\dagger = \frac{-i}{\sqrt{2\omega}}(p_j + i\omega x_j), \quad (2.16)$$

where $p_j \equiv -i\frac{\partial}{\partial x_j}$. These operators satisfy the standard bosonic commutation relation: $[a_i, a_j] = 0$, $[a_i^\dagger, a_j^\dagger] = 0$, $[a_i, a_j^\dagger] = \delta_{ij}$, for all $i, j \in \{1, 2, \dots, N\}$. In terms of these creation and annihilation operators, the number operator for the j -th oscillator is defined as

$$n_j \equiv a_j^\dagger a_j = \frac{1}{2\omega}(p_j^2 + \omega^2 x_j^2) - \frac{1}{2}, \quad (2.17)$$

and H_{QHO} in Eq.(2.13) can be expressed as

$$H_{QHO} = \omega \sum_{j=1}^N n_j. \quad (2.18)$$

Since the number operators (n_i 's) are mutually commuting conserved quantities for H_{QHO} , corresponding simultaneous eigenfunctions are given by

$$|\lambda_1, \lambda_2, \dots, \lambda_N\rangle = \prod_{j=1}^N (a_j^\dagger)^{\lambda_j} |0\rangle, \quad (2.19)$$

where λ_j ($\in \mathbb{Z}^{\geq 0}$) is the quantum number associated with the number operator n_j and $a_j|0\rangle = 0$ for all values of j . Due to the existence of the similarity transformation (2.14), one may naively think that the wave functions defined as

$$|\phi_{\lambda_1, \lambda_2, \dots, \lambda_N}\rangle \equiv T|\lambda_1, \lambda_2, \dots, \lambda_N\rangle, \quad (2.20)$$

would be eigenfunctions of \tilde{H}_D with eigenvalue $E_{\lambda_1, \lambda_2, \dots, \lambda_N} = \omega \sum_{j=1}^N \lambda_j - \frac{1}{2}N\omega$. However, before reaching to this conclusion, it is important to find out the domain of the operator T by checking whether $|\phi_{\lambda_1, \lambda_2, \dots, \lambda_N}\rangle$ represents a nonsingular, square integrable wave function. To this end, we rewrite the operator T in Eq.(2.15) as $T = e^{-\frac{1}{4\omega}O_L}\chi$, where $\chi \equiv e^{\frac{1}{4\omega}\nabla^2} e^{\frac{1}{2}\omega X^2}$. Through direct calculation it can be shown that χ satisfies the relations

$$\chi (a_j^\dagger)^{\lambda_j} = (2\omega)^{\frac{\lambda_j}{2}} x_j^{\lambda_j} \chi, \quad \chi |0\rangle = 1. \quad (2.21)$$

By using these relations, we find that $|\phi_{\lambda_1, \lambda_2, \dots, \lambda_N}\rangle$ in Eq.(2.20) can be expressed as (in the coordinate representation)

$$|\phi_{\lambda_1, \lambda_2, \dots, \lambda_N}\rangle = (2\omega)^{\frac{1}{2} \sum_{j=1}^N \lambda_j} e^{-\frac{1}{4\omega}O_L} (x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}). \quad (2.22)$$

From the above equation it is evident that, $|\phi_{\lambda_1, \lambda_2, \dots, \lambda_N}\rangle$ would be a singular wave function, if the action of the Lassalle operator O_L on the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}$ leads to a singularity. By using Eq.(2.7), we get

$$\begin{aligned} O_L(x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}) &= \sum_{j=1}^N \lambda_j (\lambda_j - 1) x_1^{\lambda_1} \dots x_j^{\lambda_j-2} \dots x_N^{\lambda_N} \\ &\quad + 4\nu \sum_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j}{x_i^2 - x_j^2} \left(x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N} \right). \end{aligned} \quad (2.23)$$

Note that first term in the r.h.s. of Eq.(2.23) seems to be singular at $x_j = 0$, whenever λ_j takes the value 0 or 1. However, the presence of the coefficient $\lambda_j(\lambda_j - 1)$ within this term precludes that possibility. On the other hand, the second term in the r.h.s. of Eq.(2.23) has pair of simple poles at $x_i = x_j$ and $x_i = -x_j$. Consequently, successive action of the Lassalle operator on the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}$ yields essential singularities at these points. Due to such singularities, $|\phi_{\lambda_1, \lambda_2, \dots, \lambda_N}\rangle$ in Eq.(2.22) does not represent a square integrable wave function.

As a first step to get rid of the above mentioned singularity, one may apply the similarity transformation T on the completely symmetrized number states of QHO, as

was done earlier [29, 30, 32] both in the cases of A_{N-1} type and B_N type of Calogero models. For the sake of convenience, we consider a system consisting of two free harmonic oscillators and define the corresponding symmetrized number states as

$$|\lambda_1, \lambda_2\rangle_s \equiv \tau(|\lambda_1, \lambda_2\rangle + |\lambda_2, \lambda_1\rangle), \quad (2.24)$$

where we assume that $\lambda_1 \leq \lambda_2$, and set $\tau = 1$ for $\lambda_1 < \lambda_2$ and $\tau = 1/2$ for $\lambda_1 = \lambda_2$. Applying the similarity transformation T on such symmetrized number state and using Eq.(2.21), we obtain

$$|\phi_{\lambda_1, \lambda_2}^s\rangle \equiv T|\lambda_1, \lambda_2\rangle_s = \tau (2\omega)^{\frac{1}{2}(\lambda_1 + \lambda_2)} e^{-\frac{1}{4\omega}O_L} (x_1^{\lambda_1} x_2^{\lambda_2} + x_1^{\lambda_2} x_2^{\lambda_1}). \quad (2.25)$$

By using Eq.(2.7), one finds that

$$\begin{aligned} O_L(x_1^{\lambda_1} x_2^{\lambda_2} + x_1^{\lambda_2} x_2^{\lambda_1}) &= \lambda_1(\lambda_1 - 1)(x_1^{\lambda_1-2} x_2^{\lambda_2} + x_1^{\lambda_2} x_2^{\lambda_1-2}) \\ &+ \lambda_2(\lambda_2 - 1)(x_1^{\lambda_1} x_2^{\lambda_2-2} + x_1^{\lambda_2-2} x_2^{\lambda_1}) + 4\nu(\lambda_2 - \lambda_1)x_1^{\lambda_1} x_2^{\lambda_1} \left(\frac{x_1^{\lambda_2-\lambda_1} - x_2^{\lambda_2-\lambda_1}}{x_1^2 - x_2^2} \right). \end{aligned} \quad (2.26)$$

Note that the singularities in the r.h.s. of the above equation can be removed completely, if we restrict the value of $\lambda_2 - \lambda_1$ to be an even integer. Indeed, by setting $\lambda_2 - \lambda_1 = 2m$, where $m \in \mathbb{Z}^{\geq 0}$, and defining symmetrized polynomials like

$$\varphi_{\lambda_1, \lambda_2} = \tau (x_1^{\lambda_1} x_2^{\lambda_2} + x_1^{\lambda_2} x_2^{\lambda_1}),$$

one can express Eq.(2.26) in the form

$$O_L \varphi_{\lambda_1, \lambda_2} = \lambda_2(\lambda_2 - 1)C_{\lambda_1, \lambda_2} \varphi_{\lambda_1, \lambda_2-2} + \lambda_1(\lambda_1 - 1)\varphi_{\lambda_1-2, \lambda_2} + 8\nu m \sum_{i=1}^t \varphi_{\lambda_1+2i-2, \lambda_2-2i},$$

where $C_{\lambda_1, \lambda_2} = (1 - \delta_{\lambda_1, \lambda_2} + \delta_{\lambda_1, \lambda_2-2})$ and $t = [(m+1)/2]$, with $[x]$ denoting the integer part of x . From the r.h.s. of the above equation it is clear that, repeated actions of O_L on $\varphi_{\lambda_1, \lambda_2}$ do not produce any singularity. Consequently, $|\phi_{\lambda_1, \lambda_2}^s\rangle$ in Eq.(2.25) would represent a nonsingular and square integrable eigenfunction of H_D , provided $\lambda_2 - \lambda_1$ is taken as an even integer.

In analogy with the two particle case, as considered in Eq.(2.25), one can construct completely symmetrized states like $|\phi_{\lambda_1, \lambda_2, \dots, \lambda_N}^s\rangle$ for the general N particle case. Such construction will be discussed in the next section. Proceeding in a similar way as has been done earlier in the case of B_N model [32], it can be shown that $|\phi_{\lambda_1, \lambda_2, \dots, \lambda_N}^s\rangle$ would represent a nonsingular eigenfunction of \tilde{H}_D , provided $\lambda_j - \lambda_i$ are even integers for all $i, j \in \{1, 2, \dots, N\}$. Note that the above condition is satisfied if either all λ_i 's are even integers, i.e. of even parity, or all λ_i 's are odd integers, i.e. of odd parity. Therefore, the Hilbert space of D_N type of Calogero model (denoted by \mathcal{H}) can be decomposed as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \quad (2.27)$$

where the subspace \mathcal{H}_0 is made of states with even parity and the subspace \mathcal{H}_1 is made of states with odd parity. It should be noted that, the decomposition of the Hilbert space given in Eq.(2.27) and corresponding eigenvalues of H_D (see Eq.(3.12)) was found earlier through a completely different approach involving the D_N type of Dunkl operators [28]. However, the present approach through similarity transformation not only enables us to reproduce these results, but also leads to explicit expressions for the corresponding eigenfunctions in a simple way.

As we have mentioned earlier that, there exists a similarity transformation which maps the B_N type of Calogero model (1.3) to a system of decoupled QHO [32]. At $\rho \rightarrow 0$ limit, that similarity transformation formally reduces to the the presently considered similarity transformation T (2.15). However, it is important to observe that, the domains of these two similarity transformations do not match with each other. To verify this thing, we note that the Lassalle operator $O_L^{(B)}$ associated with the B_N type of Calogero model (1.3) is given by [32]

$$O_L^{(B)} = \sum_{i=1}^N \left(\frac{\partial^2}{\partial x_i^2} + 2\rho \frac{1}{x_i} \frac{\partial}{\partial x_i} \right) + 4\nu \sum_{1 \leq i < j \leq N} \frac{1}{(x_i^2 - x_j^2)} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right).$$

Action of this $O_L^{(B)}$ on the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}$ yields

$$\begin{aligned} O_L^{(B)}(x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}) &= \sum_{j=1}^N \{ \lambda_j(\lambda_j - 1) + 2\rho\lambda_j \} x_1^{\lambda_1} \dots x_j^{\lambda_j-2} \dots x_N^{\lambda_N} \\ &\quad + 4\nu \sum_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j}{x_i^2 - x_j^2} (x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}). \end{aligned} \quad (2.28)$$

Comparing the first terms in the r.h.s. of Eqs. (2.23) and (2.28), we find that the coefficient $\lambda_j(\lambda_j - 1)$ in the former equation is replaced by the coefficient $\lambda_j(\lambda_j - 1) + 2\rho\lambda_j$ in the latter equation. Consequently, unlike the case of D_N type of Calogero model, the first term in the r.h.s. of Eq.(2.28) picks up a singularity at $x_j = 0$ for the choice $\lambda_j = 1$. Moreover, successive action of $O_L^{(B)}$ yields this type of singularity at $x_j = 0$ for any odd value of λ_j . Thus the similarity transformation associated with the B_N type of Calogero model generates singularity while acting on the completely symmetric states of QHO with odd parity. On the other hand, all singularities appearing in Eq.(2.28) can be eliminated by acting $O_L^{(B)}$ on the completely symmetrized form of the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_N^{\lambda_N}$ and restricting all λ_i 's to be even integers [32]. Consequently, the Hilbert space of this B_N type of Calogero model can be constructed by using such completely symmetric states with even parity only.

3 Construction of Eigenfunctions

Here our aim is to construct the eigenfunctions of the Hamiltonian H_D (1.2) for the general N particle case and find out the scalar product of such eigenfunctions. To this

end, we consider a set of nonnegative integers like $\vec{\lambda} \equiv \{\lambda_1, \lambda_2, \dots, \lambda_N\}$, subject to restriction that all λ_i 's have either positive parity or negative parity and the ordering $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$. One can construct a symmetrized number state associated with $\vec{\lambda}$ as

$$|\vec{\lambda}\rangle_s \equiv \sum_{\sigma \in S_N} |\lambda_{\sigma_1}, \dots, \lambda_{\sigma_N}\rangle = \varphi_{\vec{\lambda}}(\mathbf{a}^\dagger)|0\rangle, \quad (3.1)$$

where $\varphi_{\vec{\lambda}}(\mathbf{x})$ is a completely symmetric function of \mathbf{x} defined by [32]

$$\varphi_{\vec{\lambda}}(\mathbf{x}) = \sum_{\sigma \in S_N} x_1^{\lambda_{\sigma_1}} x_2^{\lambda_{\sigma_2}} \dots x_N^{\lambda_{\sigma_N}}, \quad (3.2)$$

with $\mathbf{x} \equiv \{x_1, x_2, \dots, x_N\} \in \mathbb{R}^N$, and the summation runs over distinct permutations so that each monomial appears only once. It may be noted that, for the particular case $N = 2$, $|\vec{\lambda}\rangle_s$ in Eq.(3.1) reproduces $|\lambda_1, \lambda_2\rangle_s$ in Eq.(2.24).

Next, by using the number operators, we define a set of mutually commuting Hermitian operators like

$$P_l(\mathbf{n}) \equiv \sum_{j=1}^N n_j^l, \quad (3.3)$$

where $l \in \{1, 2, \dots, N\}$. Due to Eq.(2.18), it follows that $H_{QHO} = \omega P_1(\mathbf{n})$. Hence $P_l(\mathbf{n})$'s represent a complete set of mutually commuting conserved quantities for the QHO. It is evident that the symmetrized number states (3.1) are simultaneous eigenfunctions of these conserved quantities:

$$P_l(\mathbf{n}) |\vec{\lambda}\rangle_s = P_l(\vec{\lambda}) |\vec{\lambda}\rangle_s, \quad (3.4)$$

where $P_l(\vec{\lambda}) = \sum_{j=1}^N \lambda_j^l$. We define the dual bases for the states (3.1) as

$$\langle \vec{\lambda} |_s \equiv \langle 0 | \varphi_{\vec{\lambda}}(\mathbf{a}^\dagger), \quad (3.5)$$

where $\langle 0 |$ is defined through the relations $\langle 0 | a_j^\dagger = 0$, for all values of j . By using the bosonic commutation relations satisfied by the creation and annihilation operators, the orthogonality relations among the scalar products of the symmetrized number states may be obtained as

$$\langle \mu | \lambda \rangle_s = \delta_{\vec{\lambda}, \vec{\mu}} \langle 0 | 0 \rangle N! \prod_{j=1}^r \frac{(l_j!)^{k_j}}{k_j!}, \quad (3.6)$$

where the notation $\langle \vec{\mu} |_s \cdot |\vec{\lambda}\rangle_s \equiv \langle \mu | \lambda \rangle_s$ is used, $\vec{\lambda}$ is written in the form

$$\vec{\lambda} = \{ \overbrace{l_1, l_1, \dots, l_1}^{k_1}, \overbrace{l_2, l_2, \dots, l_2}^{k_2}, \dots, \overbrace{l_r, l_r, \dots, l_r}^{k_r} \}, \quad (3.7)$$

such that $\sum_{i=1}^r k_i = N$, and $\langle 0|0\rangle = \left(\int_{-\infty}^{\infty} e^{-\omega x^2} dx\right)^N = \left(\frac{\pi}{\omega}\right)^{\frac{N}{2}}$.

By applying the operator T (2.15) on the symmetrized number state $|\vec{\lambda}\rangle_s$ (3.1), and using the relations (2.14) and (2.21), we obtain the eigenfunctions for \tilde{H}_D as

$$|\phi_{\vec{\lambda}}^s\rangle \equiv T|\vec{\lambda}\rangle_s = (2\omega)^{\frac{1}{2}\sum_{i=1}^N \lambda_i} e^{-\frac{1}{4\omega}O_L} \varphi_{\vec{\lambda}}(\mathbf{x}), \quad (3.8)$$

with eigenvalues given by

$$\tilde{E}_{\lambda_1, \lambda_2, \dots, \lambda_N} = \omega \sum_{j=1}^N \lambda_j - \frac{1}{2}N\omega. \quad (3.9)$$

Proceeding in a similar way as has been done earlier in the case of B_N model [32], it can be shown that $|\phi_{\vec{\lambda}}^s\rangle$ in Eq.(3.8) represents nonsingular and square integrable eigenfunctions for \tilde{H}_D . Let us now define an operator \mathcal{T} as

$$\mathcal{T} \equiv \psi_0(\mathbf{x})T = \psi_0(\mathbf{x})e^{-\frac{1}{4\omega}O_L} e^{\frac{1}{4\omega}\nabla^2} e^{\frac{1}{2}\omega X^2}, \quad (3.10)$$

where $\psi_0(\mathbf{x})$ is the ‘operator form’ of the ground state wave function (2.2). By using Eqs. (2.6) and (3.8), we obtain the eigenfunctions for the original Calogero Hamiltonian H_D (1.2) as

$$|\psi_{\vec{\lambda}}^s\rangle = \mathcal{T}|\vec{\lambda}\rangle_s = (2\omega)^{\frac{1}{2}\sum_{i=1}^N \lambda_i} \psi_0(\mathbf{x}) e^{-\frac{1}{4\omega}O_L} \varphi_{\vec{\lambda}}(\mathbf{x}). \quad (3.11)$$

Subsequently, by using Eq.(2.5), we obtain the corresponding eigenvalues as

$$E_{\lambda_1, \lambda_2, \dots, \lambda_N} = \tilde{E}_{\lambda_1, \lambda_2, \dots, \lambda_N} + E_0 = \omega \sum_{j=1}^N \lambda_j + \nu N(N-1)\omega, \quad (3.12)$$

where all λ_j ’s have the same parity and they are ordered as $\lambda_1 \geq \lambda_2 \geq \dots \lambda_N \geq 0$.

In this context it should be noted that, by acting the Hamiltonian of the D_N type of Calogero model on the corresponding Coxeter invariant Polynomials, one can get the spectrum of this model in the form [25, 26]

$$E_{m_1, \dots, m_N} = \omega \sum_{j=1}^N m_j f_j + \nu N(N-1)\omega, \quad (3.13)$$

where $f_j = 2j$ for $j \in \{1, \dots, N-1\}$, $f_N = N$ and m_j ’s are arbitrary non-negative integers. To make a connection between the eigenvalue relations (3.12) and (3.13), we define a mapping between the related quantum numbers as

$$\lambda_j = 2 \sum_{i=j}^{N-1} m_i + m_N. \quad (3.14)$$

Note that this is a one-to-one mapping, whose inverse is given by $n_j = \frac{1}{2}(\lambda_j - \lambda_{j+1})$ for $j \in \{1, \dots, N-1\}$ and $n_N = \lambda_N$. Substituting Eq.(3.14) in Eq.(3.12), and interchanging the summations over i and j indices, we find that the spectra generated by Eq.(3.12) and Eq.(3.13) match exactly. It is interesting to note that, due to Eq.(3.14), the parity of m_N determines the parity of all the λ_j 's. Consequently, the eigenvalues in Eq.(3.13) with even (odd) values of m_N are associated with the eigenfunctions (3.11) corresponding to the subspace \mathcal{H}_0 (\mathcal{H}_1).

Let us now define a new set of ‘creation’ and ‘annihilation’ operators associated with the original Calogero Hamiltonian H_D (1.2) as

$$b_j^\dagger = \mathcal{T} a_j^\dagger \mathcal{T}^{-1}, \quad \tilde{b}_j = \mathcal{T} a_j \mathcal{T}^{-1}, \quad (3.15)$$

where $j \in \{1, 2, \dots, N\}$. Similar to the case of QHO, these creation and annihilation operators also satisfy the standard bosonic commutation relation:

$$[\tilde{b}_i, \tilde{b}_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0, \quad [\tilde{b}_i, b_j^\dagger] = \delta_{ij}, \quad (3.16)$$

for all $i, j \in \{1, 2, \dots, N\}$. However, it should be noted that the operator \mathcal{T} defined in Eq.(3.10) is not an unitary operator. Consequently, b_j^\dagger is no longer the adjoint operator of \tilde{b}_j . The vacuum state associated with this new type of creation and annihilation operators may be defined as

$$|0\rangle_D \equiv \mathcal{T}|0\rangle, \quad (3.17)$$

which satisfies the relations $\tilde{b}_j|0\rangle_D = 0$ for all j , and coincides with the ground state wave function (2.2) of the D_N type of Calogero model in the coordinate representation. Due to such coincidence, the normalization condition for ground state wave function of the D_N type of Calogero model [32–34] leads to a relation like

$$\langle 0|\mathcal{T}^\dagger \mathcal{T}|0\rangle = \frac{1}{\omega^{N\{\frac{1}{2}+(N-1)\nu\}}} \prod_{j=1}^N \frac{\Gamma(1+j\nu)\Gamma(\frac{1}{2}+(j-1)\nu)}{\Gamma(1+\nu)}, \quad (3.18)$$

where $\Gamma(z)$ denotes the usual gamma function. The above equation clearly shows that \mathcal{T} can not be an unitary operator. As a result, one has to be more careful for defining the dual vector corresponding to $|0\rangle_D$. Indeed, by following the usual convention, if such dual vector is defined as $\langle 0|_D = \langle 0|\mathcal{T}^\dagger$, then this dual vector would not be annihilated by the left action of the creation operators like b_j^\dagger . To bypass this problem, we define the dual vector corresponding to $|0\rangle_D$ in Eq.(3.17) as

$$\langle 0|_D \equiv \langle 0|\mathcal{T}^{-1}, \quad (3.19)$$

which satisfies the desired relations $\langle 0|_D b_j^\dagger = 0$ for all j . This type of dual vectors, defined in a rather unconventional way, will be used shortly to construct a nontrivial inner product in the Hilbert space of the D_N type of Calogero model.

Applying the relations (2.4), (2.14) and (2.18), we can express the Calogero Hamiltonian H_D (1.2) through the number operators associated with \tilde{b}_j and b_j^\dagger as

$$H_D = \omega \sum_{j=1}^N \eta_j + \nu N(N-1)\omega, \quad (3.20)$$

where $\eta_j = b_j^\dagger \tilde{b}_j$. Furthermore, by using Eqs. (3.1), (3.15) and (3.17), it is possible to rewrite the eigenfunctions (3.11) of the D_N type of Calogero model through symmetric combination of different powers of b_j^\dagger 's as

$$|\psi_{\vec{\lambda}}^s\rangle = \mathcal{T} \varphi_{\vec{\lambda}}(\mathbf{a}^\dagger) \mathcal{T}^{-1} \cdot \mathcal{T}|0\rangle = \varphi_{\vec{\lambda}}(\mathbf{b}^\dagger)|0\rangle_D. \quad (3.21)$$

Let us now define the dual vector corresponding to $|\psi_{\vec{\mu}}^s\rangle$ as

$$\langle\psi_{\vec{\mu}}^s|_D \equiv \langle 0|_D \varphi_{\vec{\mu}}(\tilde{\mathbf{b}}), \quad (3.22)$$

which leads to a new inner product between the states $|\psi_{\vec{\lambda}}^s\rangle$ and $|\psi_{\vec{\mu}}^s\rangle$:

$$\langle\langle\psi_{\vec{\mu}}^s|\psi_{\vec{\lambda}}^s\rangle\rangle \equiv \langle 0|_D \varphi_{\vec{\mu}}(\tilde{\mathbf{b}}) \varphi_{\vec{\lambda}}(\mathbf{b}^\dagger)|0\rangle_D. \quad (3.23)$$

Since b_j^\dagger is not the adjoint operator of \tilde{b}_j , and $\langle 0|_D$ is not the dual of $|0\rangle_D$ in the conventional sense, it is obvious that the inner product given in the above equation is different from the conventional Hermitian inner product. Furthermore, it should be noted that, the inner product (3.23) is also different in nature from the inner products used earlier [32] for the cases of A_{N-1} and B_N types of Calogero models, where the duals of the vacuum states were defined in the conventional sense. Using the bosonic commutation relations (3.16) and expressing $\vec{\lambda}$ in the form (3.7), we find that the inner product (3.23) can be computed as

$$\langle\langle\psi_{\vec{\mu}}^s|\psi_{\vec{\lambda}}^s\rangle\rangle = \delta_{\vec{\lambda}, \vec{\mu}} N! \langle 0|0\rangle_D \prod_{j=1}^r \frac{(l_j!)^{k_j}}{k_j!}, \quad (3.24)$$

where $\langle 0|0\rangle_D = \langle 0|0\rangle = \left(\frac{\pi}{\omega}\right)^{\frac{N}{2}}$. Thus the eigenfunctions (3.21) of D_N type of Calogero model are orthogonal to each other with respect to the inner product (3.23).

Let us now investigate whether there exists any deeper reason for the existence of nontrivial inner product (3.23), which makes the eigenfunctions (3.21) orthogonal. In the following, it will be shown that the integrable structure of D_N type of Calogero model plays a crucial role in this matter. To this end, we apply a similarity transformation on the symmetrized conserved quantities (3.3) of the QHO and construct a set of mutually commuting conserved quantities for the D_N type of Calogero Hamiltonian H_D (1.2) as

$$P_l(\eta) = \mathcal{T} P_l(\mathbf{n}) \mathcal{T}^{-1} = \sum_{j=1}^N (\eta_j)^l, \quad (3.25)$$

where $l \in \{1, 2, \dots, N\}$. Since the exponential of the Lassalle operator has entered in the definition of \mathcal{T} in Eq.(3.10), $P_l(\eta)$'s can not be expressed in general as some finite power series of the canonical variables. Moreover, due to nonunitarity of the operator \mathcal{T} , it follows from Eq.(3.25) that $P_l(\eta)$'s are not Hermitian operators in general with respect to the conventional inner product. However, acting the operator \mathcal{T} on both sides of Eq.(3.4), we obtain the relation

$$P_l(\eta) |\psi_\lambda^s\rangle = \left(\sum_{j=1}^N \lambda_j^l \right) |\psi_\lambda^s\rangle, \quad (3.26)$$

which shows that the eigenfunctions (3.21) simultaneously diagonalize all of these mutually commuting conserved operators with a set of completely real eigenvalues.

In this context, it is useful to notice that a set of quasi-Hermitian operators (denoted by A_l 's) are defined through the relations [35]

$$A_l^\dagger = \Theta A_l \Theta^{-1}, \quad (3.27)$$

where Θ is a Hermitian, positive definite operator. Combining the operator Θ and standard inner product $\langle \phi | \psi \rangle$, one can define a new inner product as

$$\langle \phi | \psi \rangle_\Theta \equiv \langle \phi | \Theta \psi \rangle, \quad (3.28)$$

where $|\phi\rangle$ and $|\psi\rangle$ are two arbitrary state vectors in the corresponding Hilbert space. It is well known that, quasi-Hermitian operators satisfying the relations (3.27) become Hermitian with respect to the new inner product defined through Eq.(3.28). Consequently, quasi-Hermitian operators yield completely real spectra and the corresponding eigenfunctions become orthogonal with respect to the inner product $\langle \phi | \psi \rangle_\Theta$ given in Eq.(3.28). Such quasi-Hermitian operators have been studied recently due to their appearance in some parity and time reversal invariant quantum systems which yield real spectra [36–39].

Interestingly, by using Eq.(3.25), we find that the adjoint of the operators $P_l(\eta)$'s can be expressed in the form (3.27) with Θ given by

$$\Theta = (\mathcal{T} \mathcal{T}^\dagger)^{-1}. \quad (3.29)$$

Hence all $P_l(\eta)$'s are quasi-Hermitian operators which, due to Eq.(3.29), can be transformed into Hermitian operators by defining an inner product like

$$\langle \phi | \psi \rangle_\Theta \equiv \langle \phi | (\mathcal{T} \mathcal{T}^\dagger)^{-1} \psi \rangle. \quad (3.30)$$

Choosing $|\psi\rangle = |\varphi_\lambda^s\rangle$, $|\phi\rangle = |\varphi_\mu^s\rangle$ and using the above definition of the inner product, we obtain

$$\langle \psi_\mu^s | \psi_\lambda^s \rangle_\Theta = \langle 0 | \mathcal{T}^\dagger \varphi_\mu(\mathbf{b}) (\mathcal{T} \mathcal{T}^\dagger)^{-1} \varphi_\lambda(\mathbf{b}^\dagger) | 0 \rangle_D, \quad (3.31)$$

where $\mathbf{b} \equiv (\mathbf{b}^\dagger)^\dagger$. Due to Eq.(3.15), it follows that

$$\varphi_{\tilde{\mu}}(\mathbf{b}) = (\mathcal{T}\mathcal{T}^\dagger)^{-1} \varphi_{\tilde{\mu}}(\tilde{\mathbf{b}}) (\mathcal{T}\mathcal{T}^\dagger).$$

Substituting the above expression to the r.h.s. of Eq.(3.31), we find that this r.h.s. exactly matches with the r.h.s. of Eq.(3.23). Consequently, we get the remarkable relation

$$\langle \psi_{\tilde{\mu}}^s | \psi_{\tilde{\lambda}}^s \rangle_{\Theta} = \langle \langle \psi_{\tilde{\mu}}^s | \psi_{\tilde{\lambda}}^s \rangle \rangle. \quad (3.32)$$

This relation clearly shows that the inner product $\langle \langle \psi_{\tilde{\mu}}^s | \psi_{\tilde{\lambda}}^s \rangle \rangle$ defined in Eq.(3.23) emerges in a natural way from the Hermiticity condition of $P_l(\eta)$'s given in Eq.(3.25), which are quasi-Hermitian operators with respect to the conventional inner product.

4 Concluding remarks

Here we solve the eigenvalue problem of the D_N type of Calogero model (1.2), by mapping it to N number of decoupled quantum harmonic oscillators (QHO) through a similarity transformation. Though this similarity transformation apparently looks like a special case of the similarity transformation which maps the B_N type of Calogero model (1.3) to a system of decoupled QHO, interestingly we find that the domains of these two similarity transformations do not match with each other. Applying the similarity transformation operator on either all even parity or all odd parity eigenfunctions of the bosonic QHO, we explicitly construct the eigenfunctions for the D_N type of Calogero model.

It turns out that these eigenfunctions for the D_N type of Calogero model are not orthogonal with respect to the conventional inner product. However, we find that their orthogonality can be established by defining a nontrivial inner product. To explore some deeper reason for the existence of such inner product, we again use the method of similarity transformation to construct a set of mutually commuting conserved quantities for the D_N type of Calogero model. Even though these conserved quantities are quasi-Hermitian operators with respect to the conventional inner product, they can be transformed to Hermitian operators by using the nontrivial inner product which we have mentioned above. Thus the integrable structure of the D_N type of Calogero model plays an important role in determining the inner product for which the corresponding eigenfunctions are orthogonal. In future, we hope to explore whether there exists any connection between the presently derived conserved quantities for the D_N type of Calogero model and the conserved quantities for this model obtained through the Lax operator approach. Moreover, the relation between the D_N type of Jack polynomials and the eigenfunctions for the D_N type of Calogero model obtained through similarity transformation may also be another interesting topic for further investigation.

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